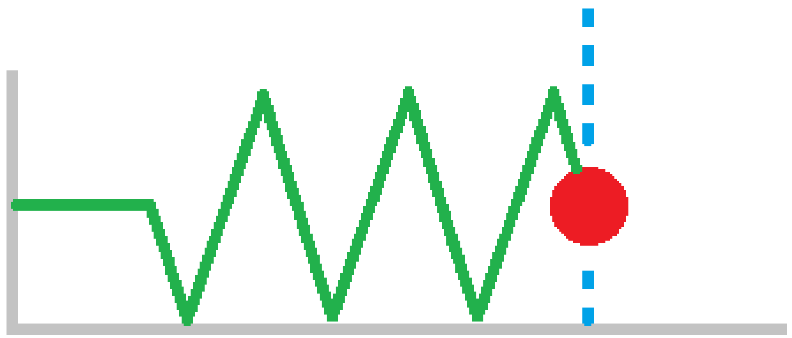
Math 270: Differential Equations Day14

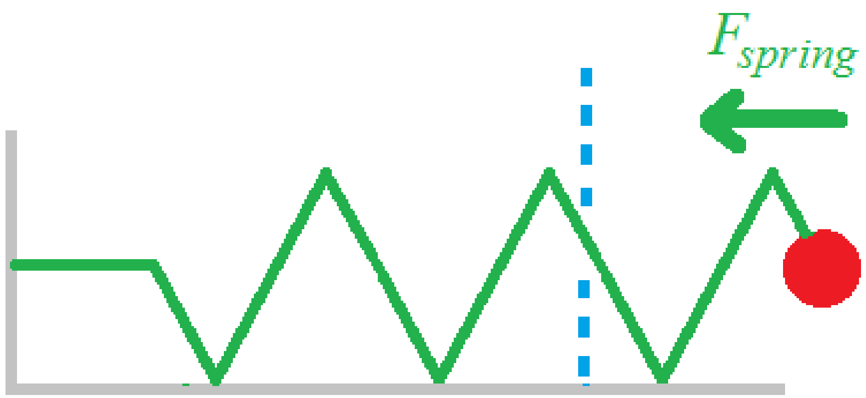
Applications of 2nd-Order Linear Differential Equations

The Mass-Spring System

Springs & Hooke's Law:

- A spring has a natural length
- If the spring is stretched beyond its natural length (length is longer than its natural length), the spring will apply a force in the direction opposite the direction it was stretched (it tries to get back to its natural length)





The Mass-Spring System

<u>Springs & Hooke's Law</u>:

- A spring has a natural length
- If the spring is compressed from its natural length (length is shorter than its natural length), the spring will apply a force in the direction opposite the direction it was compressed (it tries to get back to its natural length)

Applications of 2nd-Order Linear Differential Equations

The Mass-Spring System

Hooke's Law: If a spring is stretched (or compressed) by an amount y , then the magnitude of the force exerted by the spring is …

$$F_{spring} = ky$$

F_{spring} = the force exerted by the spring
y = the amount the spring is stretched from its natural length
k = a number, called the spring constant. (if tells you how hard it is to stretch the spring)

$$\vec{F}_{spring} = -ky$$

Applications of 2nd-Order Linear Differential Equations

The Mass-Spring System   discuss damping and external forces and most general DE

$$\vec{F}_{damping} = -by' \qquad b > 0$$

$$\vec{F}_{external} = F_{ext}(t)$$

DE: $\quad my'' + by' + ky = F_{ext}(t)$

Applications of 2nd-Order Linear Differential Equations

The Mass-Spring System (**undamped, free case**)

Derive

Applications of 2nd-Order Linear Differential Equations

The Mass-Spring System (**undamped, free case**)

Derive

DE

$y(t)$

Alternate form formulas:

Period

Frequency

Amplitude

Applications of 2nd-Order Linear Differential Equations

The Mass-Spring System (**undamped, free case**)

Example 1 A $1/8$-kg mass is attached to a spring with stiffness $k = 16$ N/m, as depicted in Figure 4.1. The mass is displaced $1/2$ m to the right of the equilibrium point and given an outward velocity (to the right) of $\sqrt{2}$ m/sec. Neglecting any damping or external forces that may be present, determine the equation of motion of the mass along with its amplitude, period, and natural frequency. How long after release does the mass pass through the equilibrium position?

Applications of 2nd-Order Linear Differential Equations

The Mass-Spring System (**damped, free case**)

Discuss

Applications of 2nd-Order Linear Differential Equations

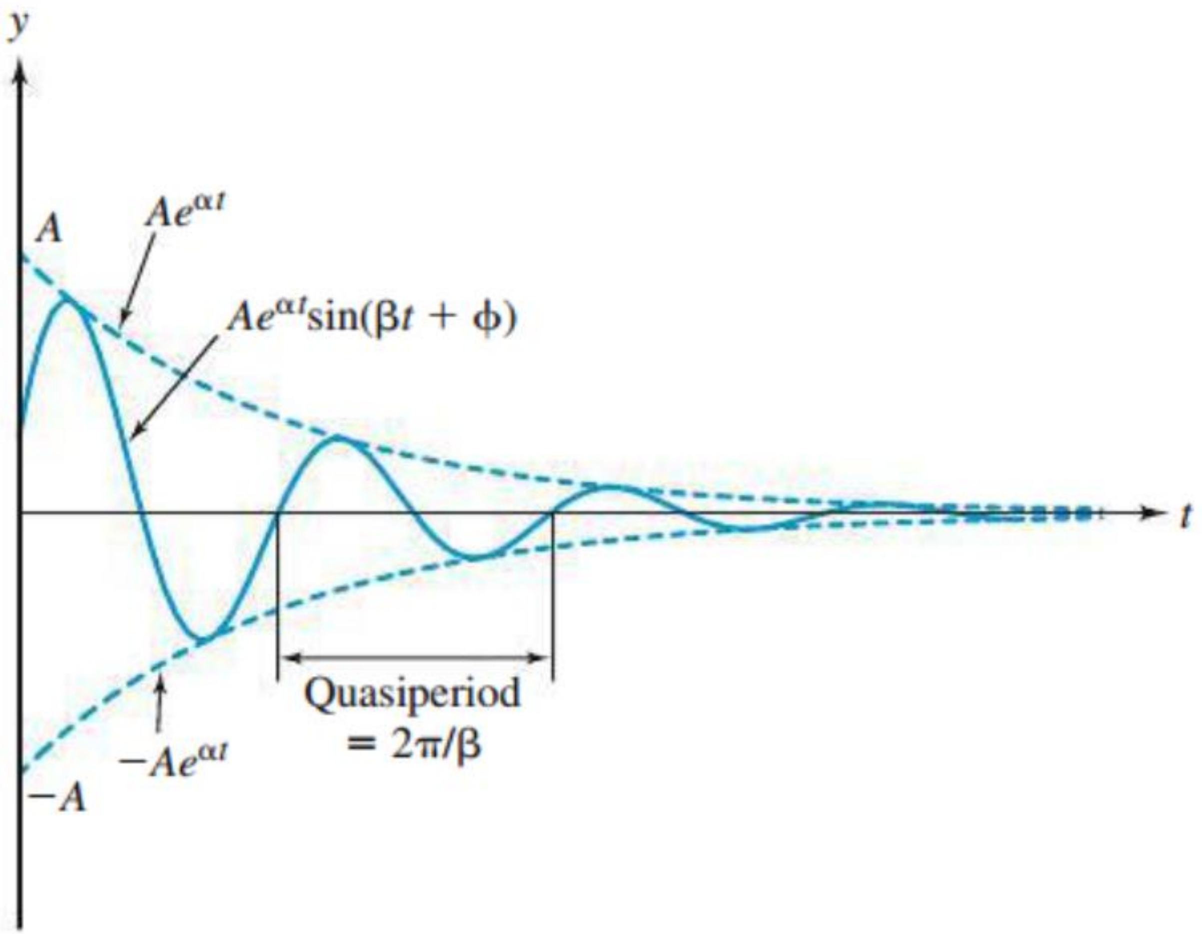
The Mass-Spring System (**damped, free case**)

Discuss Underdamped

Applications of 2^{nd}-Order Linear Differential Equations

The Mass-Spring System (**damped, free case**)

Discuss Underdamped



Applications of 2nd-Order Linear Differential Equations

The Mass-Spring System (**damped, free case**)

Discuss Overdamped

The Mass-Spring System (**damped, free case**)

Discuss Overdamped

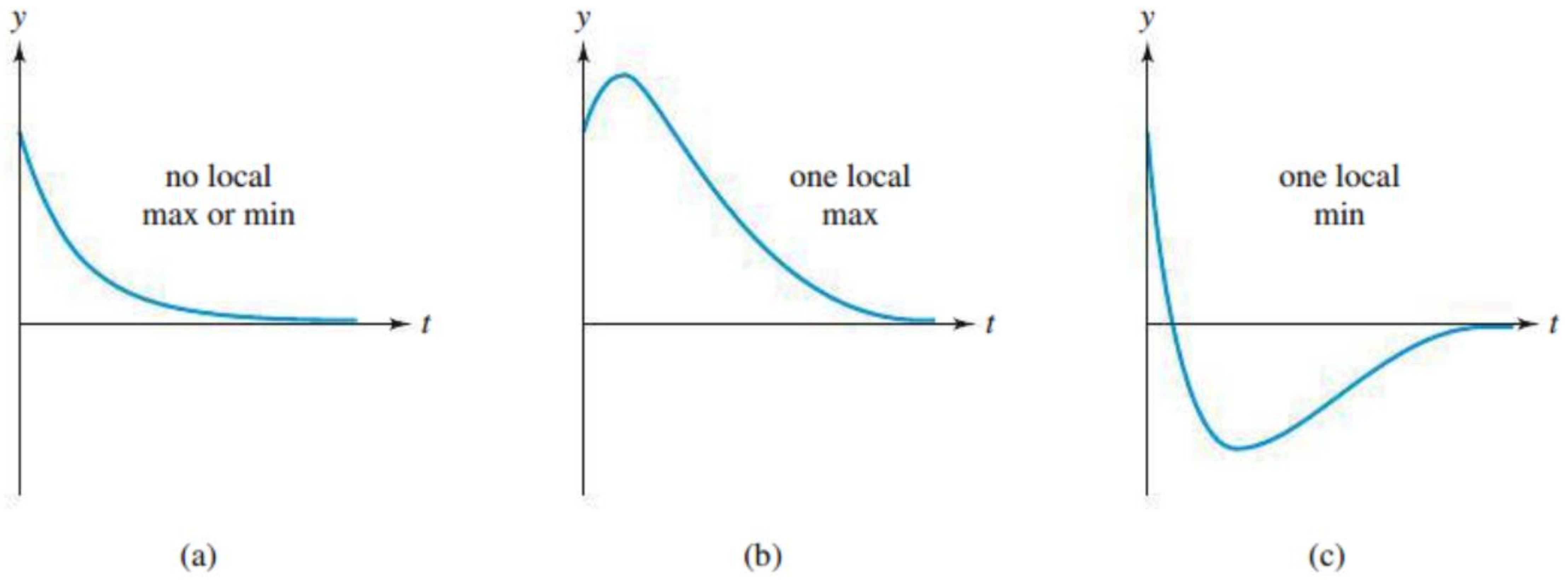


Figure 4.29 Overdamped vibrations

Applications of 2nd-Order Linear Differential Equations

The Mass-Spring System (**damped, free case**)

Discuss Critically Damped

Applications of 2nd-Order Linear Differential Equations

The Mass-Spring System (**damped, free case**)

Example 2 Assume that the motion of a mass–spring system with damping is governed by

$$\frac{d^2y}{dt^2} + b\frac{dy}{dt} + 25y = 0\,; \quad y(0) = 1\,, \quad y'(0) = 0\,.$$

Find the equation of motion and sketch its graph for the three cases where $b = 6$, 10, and 12.

Applications of 2nd-Order Linear Differential Equations

The Mass-Spring System (**damped, free case**)

Example 2

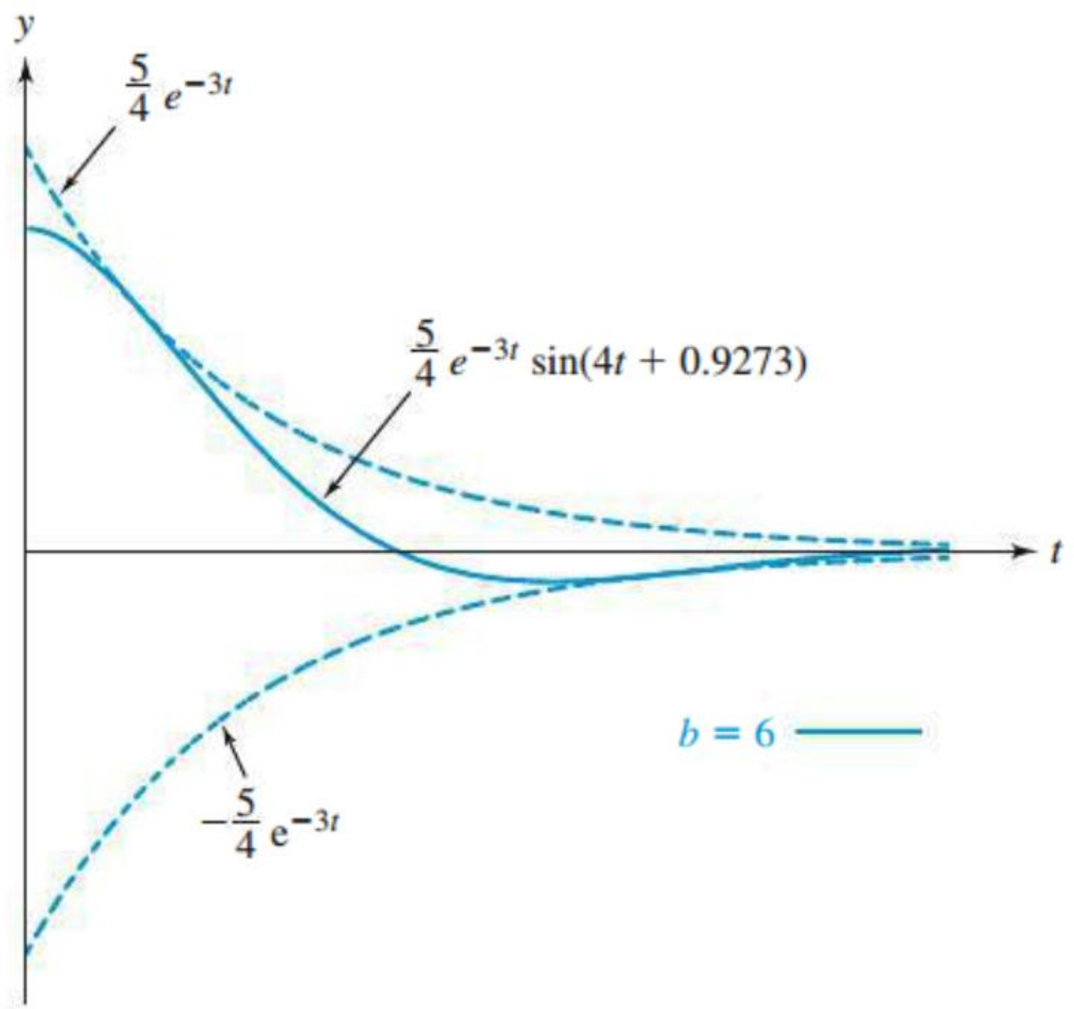
$b = 6$: $$y(t) = \frac{5}{4}e^{-3t}\sin(4t + \phi)$$

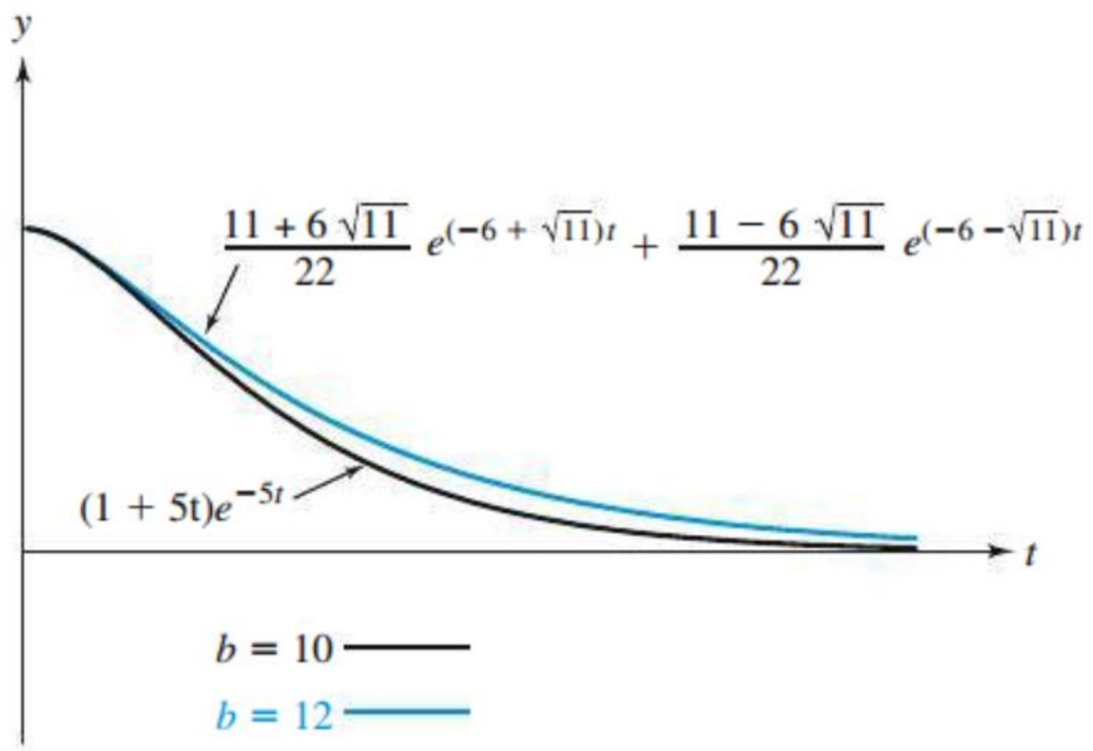
$b = 10$: $$y(t) = (1 + 5t)e^{-5t}$$

$b = 12$: $$y(t) = \frac{11 + 6\sqrt{11}}{22}e^{(-6+\sqrt{11})t} + \frac{11 - 6\sqrt{11}}{22}e^{(-6-\sqrt{11})t}$$

The Mass-Spring System (**damped, free case**)

Example 2





Applications of 2nd-Order Linear Differential Equations

The Mass-Spring System (**damped, free case**)

Example 3 A 1/4-kg mass is attached to a spring with a stiffness 4 N/m as shown in Figure 4.31(a). The damping constant b for the system is 1 N-sec/m. If the mass is displaced 1/2 m to the left and given an initial velocity of 1 m/sec to the left, find the equation of motion. What is the maximum displacement that the mass will attain?